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Mathematical Tools for Multifractal Signal Processing

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1. Introduction

Large classes of signals exhibit a very irregular behavior. In the most complicated situations this irregular behavior may follow different regimes, and can switch from one regime to another almost instantaneously. This is obviously the case for recordings of speech signals; precise recordings of turbulence data (which became available at the beginning of the 80's) show that turbulence also falls in this category. Such signals cannot be modeled by standard stationary increments processes, such as Fractional Brownian Motion (or related Gaussian processes) for instance. The techniques of multifractal signal analysis have been specifically designed to analyze such behavior. Initially developed in the mid 80's in the context of turbulence analysis, they were applied successfully to a large range of signals, including traffic data (cars and internet), stock market prices, speech signals, texture analysis, etc. We give an overview of the mathematical tools that were developed for that purpose, and we present some of the most successful applications. We start by introducing some simple mathematical tools that will be useful to model the above notions.

First, what is meant by pointwise regularity? It is a way to quantify, by using a positive real number α, the fact that the graph of a function has a certain smoothness at a point x_0.

The lowest possible level of regularity is *continuity*: A function F is continuous at x_0 if $|F(x)-F(x_0)| \to 0$ as $x \to x_0$; continuity corresponds to a regularity index $\alpha = 0$. Similarly, F is differentiable if there exists a linear function P such that $|F(x) - P(x - x_0)| \to 0$ faster than $|x - x_0|$ as $x \to x_0$; this corresponds to a regularity index $\alpha = 1$.

The following definition is a direct generalization of these two particular cases.

Let α be a positive real number and $x_0 \in \mathbb{R}^m$; a function $F : \mathbb{R}^m \to \mathbb{R}$ is $C^\alpha(x_0)$ if there exists a polynomial P of degree less than α such that

$$|F(x) - P(x - x_0)| \leq C|x - x_0|^\alpha. \tag{1}$$

Note that the constant term of $P(x - x_0)$ is $F(x_0)$, and if F is $C^{[\alpha]}$ in a neighborhood of x_0, the polynomial P is exactly the Taylor expansion of F at x_0 of order α. Nonetheless, (1) can hold for a large α even if F is not differentiable in a neighborhood of x_0 (consider, for instance, the "chirp" $x^n \sin(x^{-n})$ in a neighborhood of 0 for a large n).

The *Hölder exponent* $h(x_0)$ is the supremum of all α such that (1) holds. Note that this Hölder exponent is a function which is defined point by point and describes the local variations of the irregularity of the function F.

We are interested in analyzing signals whose Hölder exponent may widely change from point to point. This instability usually makes the task of determining the Hölder exponent

$h(x)$ very difficult numerically. In some cases the determination of the Hölder exponent is either impossible or irrelevant. This is the case for multifractal functions, where the Hölder exponent jumps from point to point. In that case the points with a given Hölder exponent form fractal sets, and one is not interested in determining the exact value of the Hölder exponent at every point but rather in extracting some relevant information concerning the size and geometry of the singularities. The relevant mathematical tool studied in this context is the *spectrum of singularities* $d(\alpha)$ (the function $d(\alpha)$ associates to each positive α the Hausdorff dimension of the set A_α of the points x where $h(x) = \alpha$). Examples of such functions include plots of random walks, interfaces developing in reaction-limited growth processes, or turbulent velocity signals at inertial range (see [3]). The most important example where one would like to determine a spectrum of singularities is probably the velocity of fully developed turbulence. The reason is that turbulent flows are not spatially homogeneous: the irregularity of the velocity seems to differ widely from point to point. This phenomenon, called "intermittency", suggests that the determination of the spectrum of singularities of the velocity of the fluid might be a nontrivial function, universal (*i.e.*, independent of boundary conditions in the limit of small viscosity), and thus would yield important information on the nature of turbulence.

Wavelet analysis has proved a powerful tool to study such classes of signals, and we will start by investigating the relationship between the pointwise regularity of a function F and decay estimates of its wavelet transform; we will state this relationship in Section 2. In Sections 3 and 4 we deal with a first application of this wavelet criterium: two solutions are given to the problem of constructing a function with a given Hölder exponent. This problem was first raised in the context of speech simulation (see [8]): a speech signal seems to have a Hölder exponent with sharp fluctuations (especially in consons). Therefore, storing its Hölder regularity might be an efficient way to compress the relevant information and could open the way to automatic speech synthesis. The first construction, in Section 3, is deterministic. It allows construction of the most general Hölder exponents. This generality must be paid for: functions constructed through the procedure that we describe are extremely peculiar and could not be used in any realistic simulation. The second construction, in Section 4, is probabilistic: we construct a 'multifractional Brownian motion' which has near the point x the same features as a fractional Brownian motion of order $h(x)$. This construction is more suited to applications, but the range of Hölder exponents that can be reached this way is smaller. In particular, $h(x)$ must be continuous.

In Section 5 we give some general results concerning multifractal functions. In particular, we show how wavelet methods give information concerning the problem of the validity of the multifractal formalism; we show that one of its limitations comes from the presence of oscillating singularities ('chirps'). We study these chirps in Section 6.

Section 7 is devoted to the study of functions that appear in many fields of applications: the functions which have a few non-vanishing wavelet coefficients (which we call, for this purpose, *lacunary wavelet series*). We show that, generically, these functions are multifractal and exhibit chirps. In Section 8 we consider Lévy processes. A Lévy process X_t ($t \geq 0$) valued in $\mathbb{R}^d$ is, by definition, a stochastic process with stationary independent increments: $X_{t+s} - X_t$ is independent of the $(X_v)_{0 \leq v \leq t}$ and has the same law as X_s. The very general definition of these processes makes them useful for modeling purposes in many fields, such as the study of stock prices. We show that they are related with lacunary wavelet series, and that they are also multifractal.

2. Pointwise regularity and wavelet coefficients

Let F be a function defined on $\mathbb{R}$. One associates with F a function of two variables, its wavelet transform defined in the "time-scale" open half plane $a > 0, b \in \mathbb{R}$ by

$$C(a,b) = \frac{1}{a} \int F(t)\psi(\frac{t-b}{a})dt \tag{2}$$

(here ψ is smooth, compactly supported, and has a vanishing integral).

Many properties of a function F can be translated into simple size estimates of its wavelet transform. If ψ is C^K and has K vanishing moments, F belongs to the Hölder space $C^\alpha(\mathbb{R})$ if (1) holds for every x_0, the constant C being independent of x_0. $F \in C^\alpha(\mathbb{R})$ (for $\alpha < K$) if and only if

$$|C(a,b)| \leq Ca^\alpha.$$

If a is small, (2) takes into account only the values of F close to b; thus it is not surprising that local properties of a function can be studied via the wavelet transform. This is, for instance, the case for pointwise regularity.

The following proposition [13] relates pointwise regularity with decay conditions of the wavelet coefficients.

PROPOSITION 1. *Let $F : \mathbb{R} \to \mathbb{R}$ be a bounded function. If F is $C^h(x_0)$,*

$$|C(a,b)| \leq Ca^h \left(1 + \frac{|b-x_0|}{a}\right)^h. \tag{3}$$

Conversely, suppose that there exists $\epsilon > 0$ such that $F \in C^\epsilon(\mathbb{R}^d)$. If (3) holds, there exists a polynomial P of degree at most $[h]$ such that, if $|x - x_0| \leq 1/2$,

$$|F(x) - P(x - x_0)| \leq C|x - x_0|^h |\log(|x - x_0|)|.$$

In order to deduce from this proposition a characterization of the Hölder exponent, let us introduce some appropriate notations. The first is a weak form of the $\mathcal{O}$ notation of Landau, and the second expresses the fact that two functions are of the same order of magnitude, disregarding "logarithmic corrections".

DEFINITION 1. *If F and G are two functions, $F = \overline{\mathcal{O}}(G)$ if*

$$\limsup \frac{\log |F|}{\log |G|} \leq 1,$$

and $F \sim G$ if

$$\lim \frac{\log |F|}{\log |G|} = 1.$$

Let h be the Hölder exponent of F at x_0. The following corollary is a direct consequence of the definition of the Hölder exponent and of Proposition 1.

COROLLARY 1. *Suppose that $F \in C^\epsilon(\mathbb{R}^d)$ for an $\epsilon > 0$. The Hölder exponent of F at x_0 is h if and only if the following two conditions hold:*

- *In the neighborhood of $(a, b) = (0, x_0)$*

$$|C(a,b)| = \overline{\mathcal{O}}(a^h + |b - x_0|^h). \tag{4}$$

- *There exists a sequence $(a_n, b_n) \to (0, x_0)$ satisfying*

$$|C(a_n, b_n)| \sim a_n^h + |b_n - x_0|^h. \tag{5}$$

We will call such a sequence (a_n, b_n) a *minimizing sequence* for F at x_0.

3. Functions with prescribed Hölder regularity

The first problem we consider is the construction of functions with a given Hölder exponent. This problem was first raised in the context of speech simulation (see [8]). A speech signal has a Hölder exponent with sharp fluctuations (especially in consons), hence the idea that a very efficient way to keep the relevant features of such a signal would be to store its Hölder exponent rather than sampled values of the signal itself.

We sketch the proof of the following theorem which completely solves, in a constructive way, the problem of determining which functions are Hölder exponents.

THEOREM 1. *A nonnegative function $h(x)$ is the Hölder exponent of a continuous function F if and only if it can be written as a lim inf of a sequence of continuous functions.*

The important point in the proof is the explicit wavelet construction of a function F which has a prescribed Hölder exponent $h(x)$. When $h(x)$ has a minimal Hölder regularity, a more natural probabilistic construction is supplied by the multifractional Brownian motion that we will study in the sequel.

Proof of Theorem 1: We let $\lambda\ (=\lambda(j,k)) = k2^{-j}$ and $\Delta_h(f) = |f(x+h) - f(x)|$. If $0 \le h(x) \le 1$,

$$h(x) = \liminf_{j\to\infty} \inf_{2^{-j} \le |h| \le 2.2^{-j}} \frac{\log(\Delta_h(f) + 2^{-j^2})}{\log h}$$

so that $h(x)$ is a lim inf of a sequence of continuous functions. The fact that this property also holds without the assumption $h(x) \le 1$ has been observed by P. Andersen [1].

Let us now prove the converse result. We suppose that $\beta(x)$ is a lim inf of a sequence of continuous functions $\beta_n(x)$. The problem is local, so that we can suppose that the β_n are uniformly continuous. Thus, there exist C^1 functions γ_n such that

$$|\gamma_n(x) - \beta_n(x)| \le C/n.$$

Let $A(n) = n + \sup|\nabla\gamma_n(x)|$.

Since it is difficult to impose the size of a wavelet transform at a given point, it is easier to use an *orthogonal wavelet basis*

$$\psi_{j,k}(x) = \psi(2^j x - k),\ j,k \in \mathbb{Z}$$

where the wavelet ψ is compactly supported, is C^K, and has a vanishing integral and K first vanishing moments, for a large enough K (see [2]). We will construct the function F by defining its wavelet coefficients on an orthonormal wavelet basis. The wavelet coefficients of F are defined as follows. If j is one of the numbers $[A(n)]$,

$$C_{j,k} = \inf(2^{-j/\log j},\ 2^{-j\gamma_n(\lambda)}),$$

otherwise we take $C_{j,k} = 0$. Let $h(x)$ be the Hölder exponent of F at x. The direct part of Proposition 1 adapts immediately to the discrete case of orthonormal wavelets (with the change of notation $a = 2^{-j}$, $b = k2^{-j}$) and obviously implies that

$$\begin{aligned} h(x) &\le \liminf_{[\lambda-2^{-j},\lambda+2^{-j}]\ni x} \gamma_n(\lambda) \\ &\le \liminf \gamma_n(x) + 2^{-j}A(n) = \liminf \beta_n(x) \end{aligned}$$

so that $h(x) \le \beta(x)$. In order to prove the converse estimate we use the second part of Proposition 1. We have

$$\begin{aligned} C_{j,k} &= \inf(2^{-j/\log j},\ 2^{-j\gamma_n(\lambda)}) \\ &\le \inf(2^{-j/\log j},\ 2^{-j\gamma_n(x)} 2^{j|x-\lambda|A(n)}). \end{aligned}$$

Since $|\lambda - x| \leq 2^{-j/(\log j)^2}$, $2^{j|x-\lambda|A(n)} \leq 2$ for n (hence j) large enough, and Proposition 1 implies that the Hölder exponent at x is exactly $\liminf \gamma_n(x)$. Let us now give as an example a class of functions that are Hölder exponents.

Let $h(x)$ be a positive measurable function in L^1. After perhaps redefining $h(x)$ on a set of measure 0, we can suppose that $\forall x$

$$h(x) = \liminf_{r\to 0} \frac{1}{Vol(B(x,r))} \int_{B(x,r)} \alpha(u)du.$$

Let $\phi(x)$ be a positive C^∞ function supported in $[-1, 1]$ of integral 1; then

$$h(x) = \liminf \alpha * \frac{1}{n}\phi(\frac{x}{n})$$

so that $h(x)$ is a Hölder exponent.

4. Multifractional Brownian Motion

This section describes some joint work with A. Benassi and D. Roux in [5] (note that an alternative construction has been proposed independently in [23]). We will construct and analyze the (one or several dimensional) *multifractional Brownian motion*. It is a stochastic process that also answers the question raised in the previous section: construct functions with given Hölder exponents. However, here the restriction on the possible exponents is more severe than before. We will have to suppose that $h(x)$ satisfies a uniform Hölder condition. The definition of the multifractional Brownian motion is a straightforward extension of the definition of the Fractional Brownian Motion of order a, except that instead of defining it by a fractional integration of order a of a white noise, we take a fractional integration of order $a(x)$ (formula (6) explains what we mean by that). We show how to construct a wavelet basis which decorrelates this process (the wavelets are no longer L^2 orthogonal but have the same localization and smoothness properties as usual wavelets: they are *vaguelettes* in the sense given by Yves Meyer in [25]). This analysis immediately yields the local regularity and scaling properties of the multifractional Brownian motion. In particular, it implies that in the neighborhood of a given point x, it is a *locally asymptotically self-similar process* of order $a(x)$.

A process X is said to be self-similar of order α if

(6) $$\forall r > 0,\ Law\ \{\ r^{-\alpha}X(rx), x \in \mathbb{R}^d\} \ = Law\ \{X(x), x \in \mathbb{R}^d\}.$$

For instance, the *Fractional Brownian Motion of order* α is self-similar (of order α). This exact scaling law can hold only for very specific processes. In order to have a more flexible notion, we define the renormalisation operators $R^{\alpha}_{x_0,r}$ by

$$R^{\alpha}_{x_0,r}X(x) = \frac{1}{r^\alpha}(X(x_0 + rx) - X(x_0)).$$

A process X is locally asymptotically self-similar (L.A.S.S.) of order $\alpha \in (0, 1)$ if there exists some non trivial limit in law for $R^{\alpha}_{x_0,r}X$ (as $r \longrightarrow \infty$).

DEFINITION 2. *Let* $\alpha(x)$ *be a function defined on* $\mathbb{R}^d$ *such that* $0 < \alpha(x) < 1$. *Let* $A = \sup \alpha(x)$ *and assume that* $\alpha(x) \in C^A(\mathbb{R}^d)$. *The multifractional Brownian motion of order* $\alpha(x)$ *is defined by*

(7) $$B_\alpha(x) = \int \frac{e^{ix.\xi} - 1}{|\xi|^{\alpha(x)+d/2}} dW(\xi).$$

Observe that this definition is a straightforward extension of the usual Fractional Brownian Motion when $\alpha(x)$ is constant.

One easily checks that the function $C : \mathbb{R}^d \to \mathbb{R}$ defined by

$$C^2(x) = \int \frac{1 - \cos^2 x.\eta}{|\eta|^{d+2\alpha(x)}} d\eta$$

belongs to $C^A(\mathbb{R}^d)$ and

$$\mathbb{E}(|B_\alpha(x+h) - B_\alpha(x)|^2) = C^2(x)|h|^{2\alpha(x)} + o(h).$$

This function will be needed in the study of the local modulus of continuity of the multifractional Brownian motion. Let us now show how to obtain a wavelet decomposition of B_α. We use the following decomposition of the white noise on the Fourier transforms of an orthonormal wavelet basis

$$dW(\xi) = \sum \xi_\lambda \hat{\psi}_\lambda(\xi) d\xi$$

where the ξ_λ are i.i.d. centered Gaussians and the wavelets are now indexed by the points $\lambda = k2^{-j}$. Let

(8) $$\omega_\lambda(x) = \int \frac{e^{ix.\xi} - 1}{|\xi|^{\alpha(x)+d/2}} \hat{\psi}_\lambda(\xi) d\xi$$

so that

(9) $$B_\alpha(x) = \sum \xi_\lambda \omega_\lambda(x).$$

The ω_λ are "vaguelettes", *i.e.*, the ω_λ and their partial derivatives satisfy the same decay estimates as the ψ_λ (see [5]).

The local regularity of the multifractional Brownian motion is given by the following theorem, which shows that, indeed, the function $\alpha(x)$ is the Hölder exponent of B_α.

THEOREM 2. *Let E be a bounded open set. Define*

$$\alpha_E = \inf_{x \in E} \alpha(x) \qquad C_E = \sup_{x \in \alpha^{-1}(\alpha_E) \cap E} c(x)$$

Law of the uniform modulus: $\mathbf{P}$ *a.e.* .

(10) $$\limsup_{|x-y| \longrightarrow 0} \frac{|B_\alpha(x) - B_\alpha(y)|}{|x-y|^{\alpha_E} \sqrt{2 \log |x-y|}} = C_E \sqrt{d}$$

Law of the iterated logarithm: $\mathbf{P}$ *a.e.* . $\forall y \in \mathbb{R}^d$,

(11) $$\limsup_{x \longrightarrow y} \frac{|B_\alpha(x) - B_\alpha(y)|}{|x-y|^{\alpha(y)} \sqrt{2 \log \log |x-y|}} = C(y).$$

Furthermore, B_α is asymptotically self similar of order $\alpha(x_0)$ at x_0; i.e.,

(12) $$\lim_{\rho \to 0^+} Law\{ \frac{B_\alpha(x_0 + \rho u) - B_\alpha(x_0)}{\rho^{\alpha(x_0)}}, u \in \mathbb{R}^d \}$$

exists for every x_0 and is not trivial.

The idea of the proof is to use the vaguelette decomposition (9), which is reminiscent, for instance, of the decomposition of the Brownian motion in the Schauder basis, and to use the decorrelation of the random variables and the localization of the ω_λ.

In [5] processes that behave locally like fractional Brownian motions are studied. Their local characteristics may change from point to point so that they are fitted to the modeling of textures that are not uniform. In [4], methods for the identification of these local parameters are proposed. These methods can be applied when only one sample path of the process is known.

5. Multifractal functions

Standard examples of functions whose Hölder exponent changes from point to point are supplied by *multifractal functions*. The relevant mathematical notion that one tries to determine when studying a multifractal function F is the *spectrum of singularities* of F, which is the function $d(h)$ defined for each $h \geq 0$ as follows: $d(h)$ is the Hausdorff dimension of the set of points x_0 where the Hölder exponent of F is h. A multifractal function has a "nontrivial" spectrum of singularities. Usually, $d(h)$ will be defined and non-constant on an interval $[h_{min}, h_{max}]$.

The examples of Sections 3 and 4, though interesting for modeling purposes in signal analysis, do not necessarily yield multifractal functions. As regards multifractal signals, it is unrealistic to determine their Hölder regularity at every point since it is usually a function that jumps everywhere. One is more interested in a qualitative approach, *i.e.*, to determine how large is the set of points which have a given Hölder exponent h Thus one wants to determine the spectrum of singularities of the signal. Frisch and Parisi introduced, in the context of fully developed turbulence, a formula which allows one to compute the spectrum of singularities of a signal by a Legendre transform of some numerically easily computable function (see [11]). Alternative formulas were proposed by Arneodo, Bacry and Muzy (see [3]). We will discuss the validity of these so-called *multifractal formalisms for functions* in several steps.

First let us mention that there exist a few mathematical examples where the Hölder exponent can be analytically determined and the validity of the multifractal formalism can be tested. These examples include the class of self-similar functions (see [18] and [6]), a number of specific examples which belong to the history of mathematics, (see [17], and references therein), and also some classes of stochastic processes, for instance, Lévy processes (see Section 8). A careful analysis of these meaningful examples gives good insight about the conditions a signal must satisfy so that the multifractal formalism is valid. Basically there must exist some self-similarity (deterministic or statistic), either of the function or of its wavelet transform. Note that in most of these examples the Hölder exponent is determined by a wavelet analysis of the function.

One of our interests is also to understand the reasons why the multifractal formalism may fail. A well-known reason is the "phase transition" phenomenon that will be explained below, but strong local oscillations of the signal can also be a more subtle cause of failure. Such strong oscillations appear when the signal has at many points a chirp-like behavior. Chirps are functions that behave like $x^\alpha \sin(1/x^\beta)$ at the origin. This phenomenon had escaped attention because the intuition concerning the multifractal formalism for functions was based on corresponding previous results concerning the multifractal formalism for measures, and of course positive measures, by definition, have no oscillations. Thus the considerations we will develop are relevant for at least two reasons:

- They show that problems posed by the multifractal formalism are of a very different nature for functions and measures.
- They give some insight about possible generalizations of the multifractal formalism for functions that would take into account this oscillatory behavior.

Such generalizations will be presented in Section 8.

If one comes back to the definition of the spectrum of singularities, it is impossible to compute numerically the spectrum of a signal since it involves the successive determination of several intricate limits, and a blind application of the formula giving the definition of the Hausdorff dimension would yield enormous, totally unstable calculations. The only method is to find some "reasonable" assumptions under which the spectrum could be derived using only averaged quantities (which should be numerically stable) extracted from the signal. Such formulas, referred to as the multifractal formalism, were inferred by physicists. Let us now state the formulas that are based on a wavelet analysis.

The Wavelet Transform Integral method: let

$$\tilde{Z}(a,q) = \int_{\mathbb{R}^m} |C(a,b)|^q db;$$

if $\tilde{Z}(a,q) \sim a^{\eta(q)}$,

$$d(\alpha) = \inf_q (q\alpha - \eta(q) + m) \tag{13}$$

The Wavelet Transform Maxima method requires first the computation of

$$Z(a,q) = \sum_{\ell} \sup_{(b,a')\in\ell} |C(a',b)|^q$$

where ℓ is a line of maxima of the wavelet transform considered on $[a,0]$, and $\sup\limits_{(b,a')\in\ell}$ means that the supremum is taken for (b,a') on this line of maxima (so that $a' \leq a$). If $Z(a,q) \sim a^{\theta(q)}$, then

$$d(\alpha) = \inf_q (qh - \theta(q)). \tag{14}$$

Numerically, according to [3], the best method seems to be the last one, probably because the restriction of the computation to the maxima insures that small errors are taken into account less (because, at the maxima, they are relatively less important). More generally, methods that involve the wavelet transform are numerically more stable, probably because they involve only averaged quantities and not the direct values of the function.

The multifractal formalism may be surprising at first sight because it relates pointwise behavior to global estimates. Before giving some mathematical justifications for it, it may be enlightening to give the heuristic argument from which it is derived. Though this argument cannot be transformed into a correct mathematical proof, it shows at least why these formulas can be expected to hold, and a careful study of its implicit assumptions shows for which type of functions it cannot hold.

Let us calculate the contribution of the Hölder singularities of order α to the integral

$$\int_{\mathbb{R}^m} |C(a,b)|^q db.$$

Near a singularity of order α we have, in a small box of size a, $|C(a,b)|^q \sim a^{\alpha q}$, because Proposition 1 basically means that if F is $C^\alpha(x_0)$ and not smoother, then the order of magnitude of its wavelet transform in the cone $|b - x_0| \leq a$ is about a^α. If the dimension of these singularities is $d(\alpha)$, it means that there are about $a^{-d(\alpha)}$ such boxes, each of volume a^m, so that the total contribution to the integral is $a^{\alpha q + m - d(\alpha)}$. The real order of magnitude of the integral should be given by the largest contribution, so that

$$\eta(q) = \inf_\alpha (\alpha q + m - d(\alpha)). \tag{15}$$

This formula is not the one we are looking for, since we know $\eta(q)$ and we want $d(\alpha)$. However, if (15) holds and if d is concave (we will see that in general this assumption need not be verified, although in many cases it is), $d(\alpha)$ is recovered by an inverse Legendre transform formula which yields (13). Of course, if d is not concave one expects (13) to yield only the convex hull of the spectrum. This is the “phase transition” phenomenon that we mentioned earlier.

Though (14) does not always hold, the following upper bound is valid if F has a minimal uniform Hölder regularity (see [21]).

PROPOSITION 2. *The following upper bound holds for any function $F \in C^{\epsilon}(\mathbb{R}^m)$ for some $\epsilon > 0$:*

(16)
$$d(\alpha) \leq \inf_p (m - \eta(p) + \alpha p).$$

In general (16) cannot be an equality, see [18]. The intuitive explanation is that any criterium which is invariant after permuting the positions of the wavelet coefficients at the same scale cannot yield the spectrum of singularities, essentially because it cannot contain the geometrical information which is relevant in the definition of the dimension. Thus, formulas involving these quantities can be true only in some very specific cases where we have an a-priori knowledge of this geometry, which is clearly the case for self-similar functions.

Let us now try to understand reasons for the failure of the multifractal formalism. The main argument in the heuristic calculation that justified it is that we interpreted Proposition 1 as meaning that if F is $C^{\alpha}(x_0)$ and not smoother, then the order of magnitude of its wavelet transform in the cone $|b - x_0| \leq a$ is about a^{α}. This holds for cusp-like singularities like $|x - x_0|^{\alpha}$ but not for "chirp-like" singularities, which have a very small wavelet transform in the cone. We will come back to this problem and study chirp behaviors carefully in the following section. We will develop in Section 8 a multifractal formalism which takes into account these oscillatory behaviors, but we first need to develop some mathematical tools that allow the capture of these chirps.

6. Chirps and oscillating singularities

This section describes joint work with Y. Meyer, A. Arneodo, E. Bacry and J-F. Muzy. Up to now we described the singularity of $f(x)$ at x_0 only by the Hölder exponent $h(x_0)$, so that, if $h(x_0) \leq 1$, we looked for the order of magnitude of $|f(x) - f(x_0)|$ when x tends to x_0, without taking into account the oscillations of $f(x) - f(x_0)$. We also want to investigate very strong oscillatory behavior (chirps of the form $x^{\alpha} \sin(1/x^{\beta})$) and show that this behavior can be characterized by simple conditions on the wavelet transform.

First we discuss the mathematical notion of *oscillating singularity* (or chirp). There is general agreement on an informal definition. A chirp of type (h, β) at x_0 should oscillate as

(17)
$$f(x) = |x - x_0|^h \sin\left(\frac{1}{|x - x_0|^{\beta}}\right)$$

in the neighborhood of x_0. Here h is the Hölder exponent of f at x_0, and β measures the speed at which the oscillations pile up near x_0 (one can relate β to the speed of divergence of the instantaneous frequency). In signal analysis, this notion covers functions whose *instantaneous frequency* increases fast at some time (see [27]). A remarkable property of (17) is that, after each integration, the Hölder exponent of the primitive at x_0 is increased by $1 + \beta$ and not 1 as could be expected. This remark is the starting point of the definition of chirps given by Yves Meyer (see [21], for instance). We will discuss the mathematical definition of an oscillating singularity. We will actually use a slightly different definition, which has the additional advantage of being stable with respect to the addition of "smooth noise", a feature which is mandatory for using this notion in signal processing. Our definition is based on the computation of pointwise Hölder exponents. We will thus prove some general results concerning Hölder exponents. This will enable us to associate to any function $F : \mathbb{R}^d \to \mathbb{R}$ two *oscillating singularity exponents* (h, β) at every point x_0.

6.1. *Oscillating singularity exponents*

The choice of the definition we use is based on the following remark. Let B be a Brownian motion and consider the function

(18) $$C(x) = x^{1/3}\sin(1/x) + B(x).$$

The strongest singularity at 0 is the chirp $x^{1/3}\sin(1/x)$, and one actually observes this oscillatory behavior after magnifying the graph sufficiently near the origin. But, after one integration, the random term becomes dominant, and the oscillatory behavior disappears. These oscillations, which do not exist in the primitive, should be reflected in the chirp exponents.

Let us now show how we can introduce a definition of *oscillating singularities* which agrees with the definition of a chirp for functions such as (17) and which has the required stability properties with respect to the addition of "smooth noise". If

(19) $$F(x) = |x-x_0|^h g(\frac{1}{|x-x_0|^\beta}) + O(|x-x_0|^{h'})$$

where $h' > h$, the first term describes the local behavior of F near x_0. In that case, we would like to say that the type of oscillating singularity at x_0 is (h,β), *i.e.*, the oscillation of F should be the oscillation of the lowest order term of its expansion. Clearly, we need to require that g has one vanishing moment. After one integration, the main term of the primitive of F may be the remaining term, which is $O(|x-x_0|^{h'+1})$. This last remark shows that, in sharp contrast with the definition of chirps given in [21], the oscillation exponent β of an oscillating singularity should not be determined using a large number of integrations, or even one integration.

Let $h^t(x_0)$ denote the Hölder exponent of the fractional primitive of order t at x_0 of the function F defined by (19). More precisely, if F is a bounded function, we denote by $h^t(x_0)$ the Hölder exponent at x_0 of the function

(20) $$F_t = (Id-\Delta)^{-t/2}(\phi F).$$

Here ϕ is a C^∞ compactly supported function satisfying $\phi(x_0) = 1$, and the operator $(Id-\Delta)^{-t/2}$ is the convolution operator which amounts to multiplying the Fourier transform of the function with $(1+|\xi|^2)^{-t/2}$. If one performs a fractional integration of order t small enough, *i.e.*, such that

(21) $$h + (1+\beta)t < h' + t,$$

then $h^t(x_0) = h + (1+\beta)t$ (here h, h' and β are the exponents used in 19). We see that the gain of pointwise Hölder regularity at x_0 after a fractional integration of very small order t is $(1+\beta)t$. One can actually show that the function $t \to h^t(x_0)$ is concave, so that its right derivative exists at 0. Hence we have the following definition for exponents of oscillating singularities.

DEFINITION 3. *Let $F : \mathbb{R}^d \to \mathbb{R}$ be a bounded function. The oscillating singularity exponents of F at a point x_0 are defined by*

(22) $$(h,\beta) = \left(h(x_0),\, \frac{\partial}{\partial t} h^t(x_0)\Big|_{t=0} - 1 \right).$$

These exponents belong to $[0,+\infty]\times[0,+\infty]$. Note that, if $h^t(x_0) = +\infty$ the exponent β is not defined.

Definition 3 shows that the exponent of oscillating singularity is defined through Hölder exponents. It should therefore be computable using the wavelet transform, as a consequence of Corollary 1. The following proposition gives a characterization of the oscillating singularity exponents at x_0 of a function $F \in C^\epsilon$. It is a direct consequence of the previous results.

PROPOSITION 3. *Let $F \in C^{\epsilon}(\mathbb{R}^d)$ for some $\epsilon > 0$. The oscillating singularity exponents at x_0 are (h, β) if and only if the wavelet transform of F satisfies the following conditions:*

- $|C(a,b)| = \overline{\mathcal{O}}(a^h + |b - x_0|^h)$ *in the neighborhood of* $(a,b) = (0, x_0)$
- *there exists a sequence* $(a_n, b_n) \to (0, x_0)$ *such that*

$$\left.\begin{array}{l} (a_n + |b_n - x_0|)^{1+\beta} \sim a_n \\ |C(a_n, b_n)| \sim a_n^h + |b_n - x_0|^h \end{array}\right\} \tag{23}$$

- β *is the smallest number such that (23) holds.*

We denote by $E_{h,\beta}$ the set of points where the oscillating singularity exponents are (h, β), and by $d(h, \beta)$ the Hausdorff dimension of $E_{h,\beta}$. By definition $d(h, \beta)$ is the *spectrum of oscillating singularities*.

7. Lacunary wavelet series

The success of wavelet techniques in many fields of application is largely due to the following remarkable property. Many signals, images, or mathematical functions can be accurately represented in a wavelet basis using very few nonzero coefficients. This is the case for piecewise smooth signals, for the velocity of turbulent flows [3], and for solutions of nonlinear hyperbolic equations (see [9]). The starting point of the denoising algorithm of images called *wavelet shrinkage* is based on the remark that, since an image composed of piecewise smooth parts has few nonzero wavelet coefficients, a noisy image can be denoised by setting to zero all small wavelet coefficients; this amounts to approximating the noisy image by a lacunary wavelet series [10].

Mathematically, the fact that a function has few non-vanishing wavelet coefficients can be formalized by stating that it belongs to Besov spaces $B_p^{s,p}$ for p close to zero (see [9]).

Though many signals or functions have been shown to be accurately represented by lacunary wavelet series, the properties of these functions have never been investigated. Our purpose is to investigate the properties of a simple probabilistic model of such series. We will see that, though the model is itself extremely simple, the corresponding random functions have an extremely complicated local structure. Namely, they exhibit a whole range of oscillating singularities located on random fractal sets.

The functions considered in the previous section have very few non-vanishing wavelet coefficients. They are thus examples of lacunary wavelet series. One might believe that the oscillating singularities are created because of the very specific position of the non-vanishing wavelet coefficients. This is in fact not the case, as the following example will show.

Let $\eta < 1$. For each $j \geq 0$ we choose at random and independently $[2^{\eta j}]$ locations $k2^{-j} \in [0, 1]$, and the corresponding wavelet coefficients $C_{j,k}$ take the value $2^{-\alpha j}$. These choices are made independently for each j. We set to 0 all other wavelet coefficients. This is the most elementary model of random lacunary wavelet series one can think of.

In order to study theses wavelet series, we now introduce the notion of *chirp exponents*, which differs slightly from the definition of "oscillating singularity exponent" given in the previous section.

DEFINITION 4. *Let $h \geq 0$ and $\beta > 0$. A function $f \in L^\infty$ is a chirp of type (h, β) at x_0 if the iterated primitives $f^{(-1)}, \dots f^{(-n)}, \dots$ satisfy $f^{(-n)} \in C^{h+n(\beta+1)}(x_0)$.*

The following characterization of [21] holds.

PROPOSITION 4. *A function $f \in L^\infty$ is a chirp of type (h, β) at x_0 if and only if there exists a function $r(x) \in C^\infty$ in a neighborhood of x_0, and $\epsilon > 0$, such that*

$$f(x) = r(x - x_0) + (x - x_0)^h g_+\left((x - x_0)^{-\beta}\right) \quad \textit{if} \quad 0 < x < \epsilon$$

and

$$f(x) = r(x - x_0) + |x - x_0|^h g_-\left(|x - x_0|^{-\beta}\right) \quad \textit{if} \quad -\epsilon < x < 0,$$

the functions g_+ and g_- being "infinitely oscillating", i.e., they have bounded primitives of any order.

The following property allows one to define "chirp exponents".

LEMMA 1. *Let $f \in L^\infty$. If f is a chirp of type (h, β) at x_0 with $\beta > 0$ and $f \in C^{h'}(x_0)$ for $h' > h$, then $\forall \beta' < \beta$, f is a chirp of type (h', β') at x_0.*

This is a straightforward consequence of Definition 4 and of the characterization given by Proposition 1. This lemma means that the interior of the set of couples (h, β) such that f is a chirp of type (h, β) at x_0 is a rectangle, $h \le h_0$, $\beta < \beta_0$. We can use these two values to define chirp exponents.

DEFINITION 5. *The chirp exponents of a function f at x_0 are*

- $h(x_0)$ *(which is the Hölder exponent at x_0),*
- $\beta(x_0) = \sup\{\beta : \exists h$ *such that f is a chirp of type (h, β) at x_0*$\}$.

Using this definition, we naturally obtain the notion of *chirp spectrum*: The chirp spectrum $d(h, \beta)$ of f is the Hausdorff dimension of the set of points where f has the chirp exponents (h, β).

The reader will have noticed the difference with the notion of *spectrum of oscillating singularities* which was defined in Section 6.1. The notion of oscillating singularities is adapted to the analysis of real signals, since it is not sensitive to perturbations by smooth noise, while the notion of chirp is adapted to mathematically defined functions. Chirps were first discovered in a nontrivial mathematical function by Y.Meyer in [21]. He showed that the trigonometric series $\sum n^{-2} \sin(n^2 x)$ has a dense set of chirps of exponents $(3/2, 1)$.

The problem of prescribing chirp exponents $h(x)$ and $\beta(x)$ simultaneously is much harder than prescribing the Hölder exponent $h(x)$ alone, and we do not know which couples of functions (h, β) can be chirp exponents. Some partial results are nonetheless available:

- The function $\beta(x)$ must vanish on a dense set (see [12]).
- Arbitrary couples (h, β) which are liminf of sequences of continuous functions can be prescribed outside some specific dense sets of measure 0, see [20].

THEOREM 3. *The lacunary wavelet series F defined above is almost surely a multifractal function. Its chirp spectrum is supported by the segment*

$$h = \alpha(\beta + 1) \quad \textit{for} \quad h \in [\alpha, \alpha/\eta]$$

and on this segment

$$d(h, \beta) = \eta(\beta + 1).$$

We won't prove this theorem here, but show how it reduces to a problem of random arcs on the circle (see [14] for a complete proof).

For each j denote by E_j the set of k's such that $C_{j,k}$ is not vanishing. Let $\delta \in [0, 1]$. Denote by $I_{j,k}$ the interval centered at $k2^{-j}$ ($k \in E_j$) and of length $2^{-\delta j}$. Let

$$E_\delta = \limsup_{j \to \infty} \bigcup_k I_{j,k} \quad \text{if} \quad \delta \in [\eta, 1),$$

$$G_\delta = \bigcap_{\delta' < \delta} E_{\delta'} - \bigcup_{\delta' > \delta} E_{\delta'} \quad \text{if} \quad \eta < \delta < 1,$$

$$G_\eta = E_\eta - \bigcup_{\delta' > \delta} E_{\delta'}, \quad G_1 = \bigcap_{\delta' < 1} E_{\delta'}.$$

The E_δ are decreasing (in δ) and E_η is almost surely the whole circle if C is large enough (this is a straightforward consequence of results on coverings of the circle by random arcs, see [22]). Thus, every point of the circle belongs to one of the G_δ. The following proposition shows that the points that belong to one of the G_δ have the same regularity and oscillation.

PROPOSITION 5. *If $x \in G_\delta$, the chirp exponents of F at x are*

$$(h,\beta) = (\frac{\alpha}{\delta}, \frac{1}{\delta} - 1).$$

(Sketch of) proof: If $x \in E_\delta$, there exists an infinity of points $k2^{-j}$ ($k \in E_j$) such that

$$|x - k2^{-j}| \le 2^{-\delta j},$$

so that the Hölder exponent of F at x is at most α/δ. By a similar argument, the Hölder exponent of $F^{(-n)}$ at x is at most $(\alpha+n)/\delta$.

Conversely, if $x \notin E_\delta$, inside a domain $|x - k2^{-j}| \le 2^{-\delta j}$ all wavelet coefficients of F vanish for j large enough. Thus, if $C_{j,k}$ is a non-vanishing wavelet coefficient,

$$C_{j,k} = 2^{-\alpha j} = (2^{-\delta j})^{\alpha/\delta} \le |x - k2^{-j}|^{\alpha/\delta},$$

which yields the proposition, and the determination of the spectrum of F is reduced to the computation of the dimensions of the sets G_δ.

8. A multifractal formalism for chirps

We use, as above, an orthogonal wavelet, and we choose L^∞ normalization for wavelets, so that we write

$$f(x) = \sum_{j,k} C_{j,k}\psi(2^j x - k) \tag{24}$$

where $C_{j,k} = 2^j \int f(t)\psi(2^j t - k)dt$. From now on we will use the following simpler notation: λ and λ' will denote respectively the intervals $\lambda_{j,k} = k2^{-j} + [0, 2^{-j}]$ and $\lambda_{j',k'} = k'2^{-j'} + [0, 2^{-j'}]$, C_λ will denote the coefficient $C_{j,k}$, and ψ_λ will denote the wavelet $\psi(2^j x - k)$.

The following new functional spaces will allow us to establish this multifractal formalism.

DEFINITION 6. *Let $p > 0$, and $s, s' \in \mathbb{R}$. A function f belongs to $\mathcal{O}_p^{s,s'}(\mathbb{R}^d)$ if its wavelet coefficients satisfy*

$$\sup_{j\in\mathbb{Z}} 2^{sj} \left(\sum_k \sup_{\lambda' \subset \lambda} |C_{\lambda'} 2^{s'j'}|^p \right)^{1/p} < \infty. \tag{25}$$

The left hand-side defines the $\mathcal{O}_p^{s,s'}(\mathbb{R}^d)$-seminorm.

Remarks: This definition is independent of the wavelet basis chosen, see [19]. It is a variation on the wavelet definition of Besov spaces, since the Besov spaces $B_p^{s,\infty}$ can be defined by the condition

$$\sup_{j\in\mathbb{Z}} 2^{-nj/p} \left(\sum_k |C_\lambda 2^{js}|^p \right)^{1/p} < \infty.$$

We now establish, by thermodynamic arguments (similar to those advocated above), a multifractal formalism which yields $d(h,\beta)$ from the knowledge of the wavelet coefficients of f. This formula is motivated by two considerations: to obtain more complete information about the Hölder singularities of the signal and, by taking into account the “chirp-type” behavior explicitly in the construction of the multifractal formalism, to eliminate one cause of failure of the standard multifractal formalism, and therefore to obtain a formula with a wider range of validity.

Let

$$\zeta(p,s') = \liminf_{j\to+\infty} \frac{\log\left(\sum_k \sup_{\lambda'\subset\lambda} |C_{\lambda'}|^p 2^{s'j'}\right)}{\log 2^{-j}} = \sup\left\{s:\ f\in \mathcal{O}_p^{s/p,s'/p}\right\}.$$

If f has a chirp of exponents (h,β) at x_0, its wavelet coefficients are of the order of magnitude of $|k2^{-j}-x_0|^h$ near the curve $2^{-j}\sim|k2^{-j}-x_0|^{1+\beta}$ and decay quickly away from this curve. Our purpose is to estimate, for each (h,β), the contribution of the chirps of exponents (h,β) to the quantity

$$\sum_{\lambda\in\Lambda_j}\sup_{\lambda'\subset\lambda}|C_{\lambda'}|^p 2^{s'j'}. \tag{26}$$

Consider a cube λ of size 2^{-j} which contains a chirp of exponents (h,β). The wavelet coefficients $C_{\lambda'}$ for $\lambda'\subset\lambda$ are negligible as long as $2^{-j'}\geq(2^{-j})^{1+\beta}$, *i.e.*, as long as $j'\leq j(1+\beta)$. When $j'\sim j(1+\beta)$, for some values of k',

$$|C_{\lambda'}|\sim(2^{-j})^h\sim 2^{-j'\frac{h}{1+\beta}},$$

so that

$$\sup_{\lambda'\subset\lambda}\left(|C_{\lambda'}|^p 2^{s'j'}\right)\sim 2^{-j'(\frac{hp}{1+\beta}-s')}\sim 2^{-j(hp-(1+\beta)s')}$$

(as long as $s'\leq ph/(1+\beta)$, otherwise the supremum is infinite). The contribution of the chirps of exponents (h,β) to (26) is thus

$$2^{d(h,\beta)j}2^{-j(hp-(1+\beta)s')}=2^{-j(hp-(1+\beta)s'-d(h,\beta))}.$$

When $j\to+\infty$, the main contribution is obtained for the couple (h,β) realizing the infimum of $hp-(1+\beta)s'-d(h,\beta)$. Hence, we have the heuristic formula

$$\zeta(s',p)=\inf_{h,\beta}\ hp-(1+\beta)s'-d(h,\beta).$$

If $d(h,\beta)$ is a convex function, it follows that

$$d(h,\beta)=\inf_{s',p}\ hp-(1+\beta)s'-\zeta(s',p). \tag{27}$$

One can easily check that this formula enables the recovery of the increasing part of the spectrum in the case of lacunary wavelet series. Obtaining the decreasing part of the spectrum would correspond to an infimum in (27) obtained for negative values of p, which cannot be expected from a mathematical approach of this type (in the case of the standard multifractal formalism, this problem already appeared, and was solved numerically by the use of the "Wavelet Maxima Method", see [3], or [18] for a mathematical discussion).

If one is interested only in obtaining the spectrum of singularities $d(h)$, it can be recovered from (27). For a fixed h, we expect the value of β which gives the largest contribution to (27) to yield the right dimension $d(h)$, hence we expect the spectrum of singularities to be obtained by the formula

$$d(h)=\sup_\beta d(h,\beta)=\sup_\beta\inf_{s',p}\ (1+\beta)s'+hp-\zeta(s',p). \tag{28}$$

Of course, this formula is by no means equivalent to the standard multifractal formalism, as can be checked on the example of lacunary wavelet series.

A mathematical investigation of the range of validity of these formulas, together with numerical algorithms, are now being worked out in collaboration with A. Arneodo, E. Bacry and J-F. Muzy, and will be the subject of a forthcoming paper.

9. Lévy processes

The sample paths of Lévy processes are a very simple and natural example of multifractal functions. Up to recently only very peculiar mathematical functions were known to be multifractal. The analysis of Lévy processes show that multifractality is generic in some sense. The very general definition of these processes makes them useful for modeling purposes in many fields, for instance finance [24].

A Lévy process X_t ($t \geq 0$) valued in $\mathbb{R}^d$ is, by definition, a stochastic process with stationary independent increments. $X_{t+s} - X_t$ is independent of the $(X_v)_{0 \leq v \leq t}$ and has the same law as X_s. Brownian motion and Poisson processes are examples of Lévy processes that can be qualified as *monofractal*. For instance, the Hölder exponent of the Brownian motion is everywhere $1/2$. These two examples are not typical. Most Lévy processes are multifractal. Furthermore, their spectrum of singularities depends precisely on the growth of the *Lévy measure* near the origin. Before stating our main theorem, we need to recall some basic definitions and results about Lévy processes.

The characteristic function of a Lévy process X_t (taking values in $\mathbb{R}^d$) satisfies

$$\mathbb{E}(e^{i\langle\lambda|X_t\rangle}) = e^{-t\psi(\lambda)},$$

where

$$\psi(\lambda) = i\langle a|\lambda\rangle + \frac{1}{2}Q(\lambda) + \int_{\mathbb{R}^d} \left(1 - e^{i\langle\lambda|x\rangle} + i\langle\lambda|x\rangle 1_{|x|<1}\right) \pi(dx).$$

Q is a positive quadratic form, and $\pi(dx)$ is the Lévy measure of X_t, *i.e.*, a positive measure defined on $\mathbb{R}^d - \{0\}$ satisfying

$$\int \inf(1, |x|^2)\pi(dx) < \infty. \tag{29}$$

The Lévy measure is usually not integrable in the neighborhood of the origin. This is, in particular, the case for stable Lévy processes of index α which satisfy (in polar coordinates)

$$\pi(dr, d\theta) = r^{-\alpha-1} dr \nu(d\theta)$$

where ν is a finite measure on the unit sphere.

When $\pi(\mathbb{R}^d) = +\infty$, the growth of the Lévy measure near the origin can be estimated using the exponent

$$\alpha = \inf\{\alpha \geq 0 : \int_{|x|\leq 1} |x|^\alpha \pi(dx) < \infty\}.$$

This exponent was shown to give the Hölder regularity of Lévy processes (without Brownian component) at $t = 0$ by W.Pruitt in [26]. The condition $\int \inf(1, |x|^2)\pi(dx) < \infty$ satisfied by Lévy measures implies that $0 \leq \alpha \leq 2$, and when X_t is a stable process, this definition coincides with the definition of the stability index.

Let

$$\begin{aligned} d_\alpha(h) &= \alpha h \quad \text{if} \quad h \in [0, 1/\alpha] \\ &= -\infty \quad \text{otherwise;} \end{aligned}$$

$$\begin{aligned} \overline{d_\alpha}(h) &= \alpha h \quad \text{if} \quad h \in [0, 1/2] \\ &= 1 \quad \text{if} \quad h = 1/2 \\ &= -\infty \quad \text{otherwise.} \end{aligned}$$

We also define

$$C_j = \int_{2^{-j} \leq |x| \leq 2.2^{-j}} \pi(dx).$$

Note that the exponent α can also be defined using the C_j's, by

$$\alpha = \sup(0, \limsup_{j\to\infty} \frac{\log_2 C_j}{j}).$$

The following theorem yields the spectrum of singularities of Lévy processes (see [15]).

THEOREM 4. *Let X_t be a Lévy process satisfying*

$$\limsup C_j = +\infty$$

and

$$\sum 2^{-j}\sqrt{C_j \log(1+C_j)} < \infty.$$

- *If X_t has no Brownian component ($Q \equiv 0$), the spectrum of singularities of almost every sample path of X_t is $d_\alpha(h)$.*
- *If X_t has a Brownian component ($Q \not\equiv 0$), the spectrum of singularities of almost every sample path of X_t is $\overline{d_\alpha}(h)$.*

Remarks:

- All Lévy processes such that $\alpha \in (0,2)$ satisfy the assumptions of Theorem 4.
- Recall that α is the almost everywhere Hölder exponent of Lévy processes without Brownian component, see [26], which of course agrees with the theorem (case where $h = \alpha$).
- Many results have been proved concerning the fractal nature of the *range* of Lévy processes, see for instance [7].

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